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MATRIX COEFFICIENTS OF THE MIDDLE DISCRETE SERIES OF $SU(2,2)$

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1. INTRODUCTION

Among the discrete series representation of the non-compact real unitary group $SU(2,2)$ of signature $(2+,2-)$, there are representations whose Gel'fand-Kirillov dimensions are 5. Because other discrete series representations have Gel'fand-Kirillov dimension 6 (the large discrete series), or 4 (holomorphic or anti-holomorphic discrete series), and also because the $(\mathfrak{g},\mathfrak{k})$-cohomology of these representations have "Hodge type" $(2,2)$ (others (3,1), (1,3), (4,0), (0,4)), we call them *the middle discrete series.*

We determine the A-radial part of the matrix coefficients with minimal K-type of a representation belonging to the middle discrete series in this paper. It is written in terms of Gaussian hypergeometric series (Main Theorem 5.5). Our method of proof is a direct computation of the A-radial part of the Schmid operator, a gradient-type operator which characterize the minimal K-type vectors in the representation space of a discrete series representation (§3). The obtained operators constitute a holonomic system of 2 variables with rank 2 (§4). It is rather complicated difference-differential equations. We honestly solve this system step by step.

2. THE GROUP $SU(2,2)$ AND ITS DISCRETE SERIES

2.1. **Structure of $SU(2,2)$ and its Lie algebra.** Let G be the special unitary group $SU(2,2)$ realized as

$$G=\{g\in SL_4(\mathbb{C})\mid g^*I_{2,2}\,g=I_{2,2}\},\qquad I_{2,2}=\mathrm{diag}(1,1,-1,-1),$$

where $g^*={}^t\overline{g}$ denotes the adjoint of a matrix g. Let $U(n)$ be the unitary group of degree n. Take a maximal compact subgroup $K=G\cap U(4)=S(U(2)\times U(2))$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebra of G, K, respectively. Let $\theta(X)=-{}^t\overline{X}$ be a Cartan involution and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$.

We set $\mathfrak{a}=\mathbb{R}H_1+\mathbb{R}H_2$ with $H_1=X_{23}+X_{32}$, $H_2=X_{14}+X_{41}$, where the X_{ij}'s are elementary matrices given by

$$X_{ij}=(\delta_{ip}\delta_{jq})_{1\le p,q\le 4}\quad\text{with Kronecker's delta }\delta_{ip}.$$

Then $\mathfrak{a}$ is a maximally $\mathbb{R}$-split abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Then the restricted root system $\Delta=\Delta(\mathfrak{g},\mathfrak{a})$ is expressed as

$$\Delta=\Delta(\mathfrak{g},\mathfrak{a})=\{\pm\lambda_1\pm\lambda_2,\ \pm2\lambda_1,\ \pm2\lambda_2\}.$$

where λ_j is the dual of H_j. We choose a positive system Δ^+ and a fundamental system Δ_{fund} of Δ:

$$\Delta^+=\{\lambda_1\pm\lambda_2,\ 2\lambda_1,\ 2\lambda_2\},\qquad \Delta_{\text{fund}}=\{\lambda_1-\lambda_2,\ 2\lambda_2\}.$$

We also denote the corresponding nilpotent subalgebra by $\mathfrak{n} = \sum_{\beta\in\Delta^+} \mathfrak{g}_\beta$. Here $\mathfrak{g}_\beta$ is the root subspace of $\mathfrak{g}$ corresponding to $\beta \in \Delta^+$. Then one obtains an Iwasawa decomposition of $\mathfrak{g}$ and G:

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}, \quad G = NAK,$$

with $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. Now let

$$\begin{aligned}
&E_1 = H_{13} - \sqrt{-1}X_{13} + \sqrt{-1}X_{31}, \quad E_2 = H_{24} - \sqrt{-1}X_{24} + \sqrt{-1}X_{42},\\
&E_3 = 1/2\,(X_{12} - X_{21} - X_{14} + X_{23} + X_{32} - X_{41} - X_{34} + X_{43}),\\
&E_4 = \sqrt{-1}/2\,(X_{12} + X_{21} - X_{14} - X_{23} + X_{32} + X_{41} - X_{34} - X_{43}),\\
&E_5 = 1/2\,(X_{12} - X_{21} + X_{14} + X_{23} + X_{32} + X_{41} + X_{34} - X_{43}),\\
&E_6 = \sqrt{-1}/2\,(X_{12} + X_{21} + X_{14} - X_{23} + X_{32} - X_{41} + X_{34} + X_{43}),
\end{aligned}$$

where $H_{ij} = \sqrt{-1}(X_{ii} - X_{jj})$ for $1 \le i < j \le 4$. Then it is easy to see that

$$\mathfrak{g}_{2\lambda_j} = \mathbb{R}E_j \quad (j = 1, 2), \quad \mathfrak{g}_{\lambda_1+\lambda_2} = \mathbb{R}E_3 + \mathbb{R}E_4, \quad \mathfrak{g}_{\lambda_1-\lambda_2} = \mathbb{R}E_5 + \mathbb{R}E_6.$$

2.2. **Parametrization of the discrete series.** Let us now parametrize the discrete series of $SU(2,2)$. Take a compact Cartan subalgebra $\mathfrak{t}$ defined by

$$\mathfrak{t} = \mathbb{R}\sqrt{-1}h^1 + \mathbb{R}\sqrt{-1}h^2 + \mathbb{R}\sqrt{-1}I_{2,2} \quad \text{with } h^1 = X_{11} - X_{22},\ h^2 = X_{33} - X_{44}$$

and let $\mathfrak{t}_\mathbb{C}$ be its complexification. Then the absolute root system, of type A_3, is given by

$$\widetilde{\Delta} = \widetilde{\Delta}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) = \{\, [\pm 2, 0; 0], [0, \pm 2; 0], [\pm 1, \pm 1; \pm 2] \,\}.$$

where $\beta = [r, s; u]$ means $r = \beta(h^1)$, $s = \beta(h^2)$ and $u = \beta(I_{2,2})$. We write the set of compact positive roots by $\widetilde{\Delta}_c^+ = \{\, [2, 0; 0], [0, 2; 0] \,\}$ and we fix it hereafter. The Weyl group $\widetilde{W} = \widetilde{W}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ is generated by s_1, s_2, s_3 where

$$\begin{aligned}
s_1[r, s; u] &= [-r, s; u],\\
s_2[r, s; u] &= [(r - s + u)/2, (-r + s + u)/2; r + s],\\
s_3[r, s; u] &= [r, -s; u].
\end{aligned}$$

We identify $\widetilde{W}$ and the symmetric group $\mathfrak{S}_4$ of degree 4 by the map: $s_j \mapsto (j, j+1)$. The compact Weyl group is given by $\widetilde{W}_c = \langle s_1, s_3 \rangle$, also identified canonically with the subgroup $\mathfrak{S}_2 \times \mathfrak{S}_2$.

There are exactly six positive systems $\widetilde{\Delta}_\mathrm{I}^+, \widetilde{\Delta}_\mathrm{II}^+, \ldots, \widetilde{\Delta}_\mathrm{VI}^+$ containing $\widetilde{\Delta}_c^+$, defined by $\widetilde{\Delta}_J^+ = w_J \widetilde{\Delta}^+$, where

$$\widetilde{\Delta}^+ = \{\, [2, 0; 0], [0, 2; 0], [\pm 1, \pm 1; 2] \,\}$$

and the elements $w_J \in \widetilde{W}$ are given by

$$w_\mathrm{I} = 1,\ w_\mathrm{II} = s_2,\ w_\mathrm{III} = s_2 s_3,\ w_\mathrm{IV} = s_2 s_1,\ w_\mathrm{V} = s_2 s_3 s_1,\ w_\mathrm{VI} = s_2 s_1 s_3 s_2.$$

We denote by $\widetilde{\Delta}_{J,n}^+$ the noncompact positive roots in $\widetilde{\Delta}_J^+$.

By definition, the space of the Harish-Chandra parameters Ξ_c is given by

$$\Xi_c = \{\Lambda \in \mathfrak{t}_\mathbb{C}^* \mid \Lambda \text{ is } \widetilde{\Delta}\text{-regular, } K\text{-analytically integral and } \widetilde{\Delta}_c^+\text{-dominant}\}.$$

Put $\Xi_J = \{\Lambda \in \Xi_c \mid \widetilde{\Delta}_J^+\text{-dominant}\}$. We also put $\rho_{G,J} = 2^{-1}\sum_{\beta\in\widetilde{\Delta}_J^+}\beta$, $\rho_c = 2^{-1}\sum_{\beta\in\widetilde{\Delta}_c^+}\beta$ and $\rho_{J,n} = 2^{-1}\sum_{\beta\in\widetilde{\Delta}_{J,n}^+}\beta$ the half sum of positive roots, the half sum of compact positive roots and the the half sum of noncompact positive roots, respectively. The space $\Xi_c \subset \mathfrak{t}_{\mathbb{C}}^*$ are divided into six parts: $\Xi_c = \bigcup_{\mathrm{I}\le J\le \mathrm{V}}\Xi_J$. For $\Lambda \in \bigcup_{\mathrm{I}\le J\le\mathrm{VI}}\Xi_J$, we denote the corresponding discrete series by π_Λ. We say that π_Λ is the middle discrete series representation if $\Lambda \in \Xi_{\mathrm{III}}\cup\Xi_{\mathrm{IV}}$. Particularly in this case, we have

$$\Delta_{\mathrm{III}}^+ = \{[1,\pm1;\pm2]\},\ \ \rho_{\mathrm{III},n} = [2,0;0],\ \ \Delta_{\mathrm{IV}}^+ = \{[\pm1,1;\pm2]\},\ \ \rho_{\mathrm{IV},n} = [0,2;0].$$

2.3. **Representations of the maximal compact subgroup.** Let $d_1, d_2 \in \mathbb{Z}_{\ge 0}$ and $d_3 \in \mathbb{Z}$. For $d = [d_1,d_2;d_3] \in \mathfrak{t}_{\mathbb{C}}^*$, define $\tau_d \in \widehat{K}$ by the following rule ($j = 1,2$):

$$(1)\qquad \begin{aligned} \tau_d(h^j)f_{k_1k_2}^{(d)} &= (2k_j - d_j)f_{k_1k_2}^{(d)}, & \tau_d(I_{2,2})f_{k_1k_2}^{(d)} &= d_3 f_{k_1k_2}^{(d)},\\ \tau_d(e_+^j)f_{k_1k_2}^{(d)} &= (d_j - k_j)f_{k_1+\delta_{1j},k_2+\delta_{2j}}^{(d)}, & \tau_d(e_-^j)f_{k_1k_2}^{(d)} &= k_j f_{k_1-\delta_{1j},k_2-\delta_{2j}}^{(d)}. \end{aligned}$$

Here, $V_d = \{\, f_{k_1k_2}^{(d)} \mid 0 \le k_j \le d_j\,\}_{\mathbb{C}}$ is the standard basis (see [2, §3]) and

$$h^1,\quad h^2,\quad e_+^1 = X_{12},\quad e_+^2 = X_{34},\quad e_-^j = {}^te_+^j$$

are the generators of $\mathfrak{k}_{\mathbb{C}}$. Then according to [2, Prop. 3.1], $\widehat{K}$ is exhausted by

$$\{(\tau_d, V_d) \mid d = [d_1,d_2;d_3],\ d_1+d_2+d_3 \text{ is even}\}.$$

The adjoint representation $\mathrm{Ad} = \mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}$ of K on $\mathfrak{p}_{\mathbb{C}}$ is decomposed into a direct sum of two irreducible subrepresentations: $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, where,

$$\mathfrak{p}_+ = \mathbb{C}X_{13} + \mathbb{C}X_{14} + \mathbb{C}X_{23} + \mathbb{C}X_{24},\qquad \mathfrak{p}_- = {}^t\mathfrak{p}_+.$$

In fact, $\mathrm{Ad}_\pm = \mathrm{Ad}|_{\mathfrak{p}_\pm}$ is isomorphic to $\tau_{[1,1;\pm2]}$, respectively. For later use, we fix the K-isomorphisms $\iota_\pm : \mathfrak{p}_\pm \to V_{[1,1;\pm2]}$ (write $f_{kl} = f_{kl}^{[1,1;\pm2]}$):

$$\begin{aligned} \iota_+ &: (X_{23},X_{13},X_{24},X_{14}) \mapsto (f_{00},f_{10},-f_{01},-f_{11}),\\ \iota_- &: (X_{41},X_{31},X_{42},X_{32}) \mapsto (f_{00},f_{01},-f_{10},-f_{11}), \quad \text{([2, Prop. 3.10].)} \end{aligned}$$

The irreducible decomposition of $\mathfrak{t}_{\mathbb{C}}$-module $V_d\otimes\mathfrak{p}_{\mathbb{C}}$ is given as

$$V_d\otimes\mathfrak{p}_{\mathbb{C}} = V_d\otimes\mathfrak{p}_-\oplus V_d\otimes\mathfrak{p}_-,\quad V_d\otimes\mathfrak{p}_\pm \simeq \bigoplus_{\epsilon_1,\epsilon_2\in\{\pm1\}} V_{[r+\epsilon_1,s+\epsilon_2;u\pm2]}.$$

The projectors

$$P_{rs}^{(\epsilon_1,\epsilon_2)} : V_d\otimes\mathfrak{p}_+ \to V_{[r+\epsilon_1,s+\epsilon_2;u+2]},\qquad \overline{P}_{rs}^{(\epsilon_1,\epsilon_2)} : V_d\otimes\mathfrak{p}_- \to V_{[r+\epsilon_1,s+\epsilon_2;u-2]},$$

are explicitly given by [2, Lemma 3.12].

2.4. ***K*-types of the middle discrete series representations.** Let π_Λ be the discrete series representation of G with Harish-Chandra parameter $\Lambda \in \Xi_J$. Then the Blatter parameter of π_Λ becomes $\lambda = \Lambda + \rho_{G,J} - 2\rho_c$. In the following we put $d = [r,s;u]$.

If $\Lambda \in \Xi_{\mathrm{III}}$ (resp. Ξ_{IV}), then the K-types τ_λ of π_Λ are parametrized by

$$[r,s;u] \text{ with } r > s+2+|u|,\ r\in\mathbb{Z}_{>0},\ s\in\mathbb{Z}_{\ge0},\ r+s+u\in2\mathbb{Z},$$
$$(\text{resp. } [r,s;u] \text{ with } s > r+2+|u|,\ r\in\mathbb{Z}_{\ge0},\ s\in\mathbb{Z}_{>0},\ r+s+u\in2\mathbb{Z}.)$$

3. SCHMID'S DIFFERENTIAL OPERATOR

3.1. **(τ, τ^*)-matrix coefficient of the middle discrete series.** Let (π_Λ, H_Λ) be a middle discrete series representation, and (τ_d, V_d) its minimal K-type. Put $d = [r, s; u]$. Then the contragredient representation τ_d^* of τ_d is isomorphic to $\tau_{[r,s;-u]}$. We identify the representation spaces V_d, V_d^* with their unique images in H_Λ, H_Λ^* respectively. Then the matrix coefficient of π is defined by

$$\langle \pi_\Lambda(g)v, w^* \rangle$$

for $v \in V_d$, $w^* \in V_d^* \subset H_\pi^*$. Here we consider a more convenient vector-valued function:

$$\Phi_{\pi,\tau}(g) = \sum_{i,j,k,l} \langle \pi_\Lambda(g) f_{kl}^*, f_{ij}^* \rangle f_{ij} \otimes f_{kl}$$

where $\{f_{ij} = f_{ij}^{[r,s;u]}\}_{ij}$ (resp. $\{f_{kl} = f_{kl}^{[r,s;-u]}\}_{kl}$) is a standard basis of V_d (resp. V_d^*). Then we find that $\Phi_{\pi,\tau}$ belongs to the following function space:

$$C^\infty_{\tau,\tau^*}(K \backslash G/K) = \left\{\phi\colon G \to V_d \otimes V_d^* \mid \phi(k_1 g k_2) = \tau_d(k_1) \otimes \tau_d^*(k_2^{-1})\phi(g), \quad k_j \in K\right\}.$$

For simplicity, we write the index $M = (i, j; k, l)$ and coefficients $c_M(g) = \langle \pi_\Lambda(g) f_{ij}^*, f_{kl}^* \rangle$.

Due to the Cartan decomposition $G = KAK$, $c_M(g)$ is determined uniquely by its restriction to A.

Lemma 3.1. *If c_M is not zero, it satisfies the condition:*

$$k_1 + l_1 + k_2 + l_2 = r + s.$$

Proof. The centralizer of A in K is

$$\{m = \mathrm{diag}(u, \bar{u}\epsilon, u, \bar{u}\epsilon) \mid |u| = 1,\ \epsilon = \pm 1\}.$$

Therefore $\phi \in C^\infty_{\tau,\tau^*}(K \backslash G/K)$ satisfies

$$\phi(mam^{-1}) = \phi(a) \qquad a \in A, m \in Z_K(A),$$

which implies the assertion. □

We can construct two intertwing operators Φ_π^R, $\Phi_{\pi^*}^L$ using the matrix coefficients:

$$\Phi_\pi^R \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, C^\infty_{\tau_d}(K \backslash G)),$$
$$\Phi_{\pi^*}^L \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, C^\infty_{\tau_d^*}(G/K)),$$

by

$$\Phi_\pi^R(v)(g) = \sum_{ij} \langle \pi(g)v, f_{ij}^* \rangle f_{ij},$$

$$\Phi_{\pi^*}^L(w)(g) = \sum_{kl} \langle f_{kl}^*, \pi^*(g^{-1})w \rangle f_{kl}.$$

If we put

$$\Phi_{\pi,\tau}^R(g) = \sum_{kl} \Phi_\pi^R(f_{kl})(g) \otimes f_{kl},$$

$$\Phi_{\pi,\tau}^L(g) = \sum_{ij} f_{ij} \otimes \Phi_{\pi^*}^L(f_{ij})(g),$$

then $\Phi^R_{\pi,\tau}(g)$ and $\Phi^L_{\pi^*,\tau}(g)$ are identical to $\Phi_{\pi,\tau}(g)$.

3.2. **Some functions on A.** We put $a_i = \exp(t_i)$ for the element $a = \exp(t_1 H_1 + t_2 H_2)$ of the $\mathbb{R}$-split torus A, We use for notation the following symbols:

$$\operatorname{sh}(x) = (x - x^{-1})/2, \quad \operatorname{ch}(x) = (x + x^{-1})/2, \quad \operatorname{cth}(x) = \operatorname{ch}(x)/\operatorname{sh}(x),$$

$$D = D(a) = \operatorname{sh}(a_1^2) - \operatorname{sh}(a_2^2), \ \ p = p(a) = \operatorname{ch}(a_1)\operatorname{ch}(a_2), \ \ t = t(a) = (\operatorname{ch}(a_1)/\operatorname{ch}(a_2))^2,$$

$$z_\pm(t) = (\operatorname{ch}(a_1)/\operatorname{ch}(a_2) \pm \operatorname{ch}(a_2)/\operatorname{ch}(a_1)), \ \ \partial_j = a_j \frac{\partial}{\partial a_j}, \ \ \partial_t = t\frac{\partial}{\partial t}, \ \ \partial_p = p\frac{\partial}{\partial p}.$$

3.3. **The Schmid operator.** Let τ_{d_1}, τ_{d_2} be representations of K. For $F(g) \in C^\infty_{d_1,d_2}(K\backslash G/K)$ and orthonormal basis $\{X_k\}$ of $\mathfrak{p}$,

$$\nabla^R_{d_1,d_2} F(g) = \sum_k R_{X_k} F(g) \otimes X_k,$$

$$\nabla^L_{d_1,d_2} F(g) = \sum_k L_{X_k} F(g) \otimes X_k,$$

are called the Schmid operator. Here R_g, (resp. L_g) is a right (resp. left) translation. Put $\mathcal{D}^{(J),R}_{d_1,d_2} = P^{(J)}_{d_2} \circ \nabla^R_{d_1,d_2}$, $\mathcal{D}^{(J),L}_{d_1,d_2} = P^{(J)}_{d_1} \circ \nabla^L_{d_1,d_2}$ with defining the projectors

$$P^{(J)}_d : V_d \otimes \mathfrak{p}_\mathbb{C} \to V_d^- = \bigoplus_{\beta \in \tilde{\Delta}^+_{n,J}} V_{d-\beta}.$$

Theorem 3.2 ([7]). *Let $\Lambda \in \Xi_J$. Then,*

$$\operatorname{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, C^\infty_{\tau_d}(K\backslash G)) \simeq \ker(\mathcal{D}^{(J),R}_{d,d^*}),$$

$$\operatorname{Hom}_{(\mathfrak{g},K)}(\pi^*_\Lambda, C^\infty_{\tau^*_d}(G/K)) \simeq \ker(\mathcal{D}^{(J),L}_{d,d^*}),$$

where d is the Blattner parameter of π_Λ.

We see that $\nabla^{L/R}$ is also decomposed into $\nabla^{L/R}_+ + \nabla^{L/R}_-$ along the decomposition $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ + \mathfrak{p}_-$ (see [2, §6]). The following formula is found in [5]:

Theorem 3.3 (Koseki-Oda). *Let $\nabla^{R/L}_\pm$ be the Schmid operators and $\rho_A(\nabla^{R/L}_\pm)$ their restriction to A. Put $Z_{13} = 2^{-1}(I_{2,2} + h^1 - h^2)$, $Z_{24} = 2^{-1}(I_{2,2} - h^1 + h^2)$ and $\tau^{(*)}_\pm = \tau^{(*)} \otimes \mathrm{Ad}_\pm$. Then, we have*

$$\begin{aligned}
\rho_A(\nabla^R_+)\phi &= \frac{1}{2}\left(\partial_1 - \operatorname{sh}(a_1^2)^{-1}\tau(Z_{13}) - \operatorname{cth}(a_1^2)\tau^*_+(Z_{13}) + 2\operatorname{cth}(a_1^2) + \frac{2}{D}\operatorname{sh}(a_1^2)\right)(\phi \otimes X_{13}) \\
&+ \frac{1}{2}\left(\partial_2 - \operatorname{sh}(a_2^2)^{-1}\tau(Z_{24}) - \operatorname{cth}(a_2^2)\tau^*_+(Z_{24}) + 2\operatorname{cth}(a_2^2) - \frac{2}{D}\operatorname{sh}(a_2^2)\right)(\phi \otimes X_{24}) \\
&+ \frac{1}{D}\left(\operatorname{ch}(a_1)\operatorname{sh}(a_2)\tau(e^1_-) + \operatorname{sh}(a_1)\operatorname{ch}(a_2)\tau(e^2_-) + \operatorname{sh}(a_2)\operatorname{ch}(a_2)\tau^*_+(e^1_-)\right. \\
&\left. + \operatorname{sh}(a_1)\operatorname{ch}(a_1)\tau^*_+(e^2_-)\right)(\phi \otimes X_{14}) \\
&- \frac{1}{D}\left(\operatorname{sh}(a_1)\operatorname{ch}(a_2)\tau(e^1_+) + \operatorname{ch}(a_1)\operatorname{sh}(a_2)\tau(e^2_+) + \operatorname{sh}(a_1)\operatorname{ch}(a_1)\tau^*_+(e^1_+)\right. \\
&\left. + \operatorname{sh}(a_2)\operatorname{ch}(a_2)\tau^*_+(e^2_+)\right)(\phi \otimes X_{23}),
\end{aligned}$$

$$
\begin{aligned}
\rho_A(\nabla_-^R)\phi &= \frac{1}{2}\left(\partial_1 + \mathrm{sh}(a_1^2)^{-1}\tau(Z_{13}) + \mathrm{cth}(a_1^2)\tau_2^-(Z_{13}) + 2\,\mathrm{cth}(a_1^2) + \frac{2}{D}\,\mathrm{sh}(a_1^2)\right)(\phi\otimes X_{31})\\
&+ \frac{1}{2}\left(\partial_2 + \mathrm{sh}(a_2^2)^{-1}\tau(Z_{24}) + \mathrm{cth}(a_2^2)\tau_2^-(Z_{24}) + 2\,\mathrm{cth}(a_2^2) - \frac{2}{D}\,\mathrm{sh}(a_2^2)\right)(\phi\otimes X_{42})\\
&- \frac{1}{D}\left(\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau(e_+^1) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau(e_+^2) + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_2^-(e_+^1)\right.\\
&\left.+ \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_2^-(e_+^2)\right)(\phi\otimes X_{41})\\
&+ \frac{1}{D}\left(\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau(e_-^1) + \mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau(e_-^2) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_2^-(e_-^1)\right.\\
&\left.+ \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_2^-(e_-^2)\right)(\phi\otimes X_{32}),\\
\rho_A(\nabla_+^L)\phi &= -\frac{1}{2}\left(\partial_1 - \mathrm{sh}(a_1^2)^{-1}\tau^*(Z_{13}) - \mathrm{cth}(a_1^2)\tau_+(Z_{13}) + 2\,\mathrm{cth}(a_1^2) + \frac{2}{D}\,\mathrm{sh}(a_1^2)\right)(\phi\otimes X_{13})\\
&- \frac{1}{2}\left(\partial_2 - \mathrm{sh}(a_2^2)^{-1}\tau^*(Z_{24}) - \mathrm{cth}(a_2^2)\tau_+(Z_{24}) + 2\,\mathrm{cth}(a_2^2) - \frac{2}{D}\,\mathrm{sh}(a_2^2)\right)(\phi\otimes X_{24})\\
&- \frac{1}{D}\left(\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e_-^1) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e_-^2) + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_+(e_-^1)\right.\\
&\left.+ \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_+(e_-^2)\right)(\phi\otimes X_{14})\\
&+ \frac{1}{D}\left(\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e_+^1) + \mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e_+^2) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_+(e_+^1)\right.\\
&\left.+ \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_+(e_+^2)\right)(\phi\otimes X_{23}),\\
\rho_A(\nabla_-^L)\phi &= -\frac{1}{2}\left(\partial_1 + \mathrm{sh}(a_1^2)^{-1}\tau^*(Z_{13}) + \mathrm{cth}(a_1^2)\tau^-(Z_{13}) + 2\,\mathrm{cth}(a_1^2) + \frac{2}{D}\,\mathrm{sh}(a_1^2)\right)(\phi\otimes X_{31})\\
&- \frac{1}{2}\left(\partial_2 + \mathrm{sh}(a_2^2)^{-1}\tau^*(Z_{24}) + \mathrm{cth}(a_2^2)\tau^-(Z_{24}) + 2\,\mathrm{cth}(a_2^2) - \frac{2}{D}\,\mathrm{sh}(a_2^2)\right)(\phi\otimes X_{42})\\
&+ \frac{1}{D}\left(\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e_+^1) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e_+^2) + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau^-(e_+^1)\right.\\
&\left.+ \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau^-(e_+^2)\right)(\phi\otimes X_{41})\\
&- \frac{1}{D}\left(\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e_-^1) + \mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e_-^2) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau^-(e_-^1)\right.\\
&\left.+ \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau^-(e_-^2)\right)(\phi\otimes X_{32}).
\end{aligned}
$$

4. Holonomic system for the spherical functions

We treat the case of $\Lambda \in \Xi_{\mathrm{III}} \cup \Xi_{\mathrm{IV}}$. Then, the Blattner parameter of π_Λ in $\Lambda \in \Xi_{\mathrm{III}}$ (resp. $\Lambda \in \Xi_{\mathrm{IV}}$) is $d = \Lambda + [1,-1;0]$ (resp. $\Lambda + [-1,1;0]$).

Lemma 4.1. *The projector $P_d^{(\mathrm{III})}$ decomposes into four projectors as follows*:

$$
\begin{aligned}
P_d^{(\mathrm{III})} &= P^{(-,+)} \oplus P^{(-,-)} \oplus \overline{P}^{(-,+)} \oplus \overline{P}^{(-,-)},\\
P_d^{(\mathrm{IV})} &= P^{(+,-)} \oplus P^{(-,-)} \oplus \overline{P}^{(+,-)} \oplus \overline{P}^{(-,-)}.
\end{aligned}
$$

Proof. We find that

$$\widetilde{\Delta}^+_{n,\mathrm{III}} = \{[1,1;\pm 2],\ [1,-1;\pm 2]\},\ \ \widetilde{\Delta}^+_{n,\mathrm{IV}} = \{[1,1;\pm 2],\ [-1,1;\pm 2]\}$$

Thus the lemma follows. □

According to Theorem 3.2, spherical functions are characterized by the differential equations derived by the composition of the Schmid operator and projectors which appears in the decomposition of $P_d^{(\mathrm{III})}$. Let $\Phi_{\pi_\Lambda,\tau_d}(a) = \sum_M c_M(a) f_{k_1,l_1} \otimes f_{k_2,l_2}$ for $M = (k_1, l_1; k_2, l_2)$. Then c_M's satisfy the following system which is equivalent to $\mathcal{D}^{(\mathrm{III}),R/L}_{d_1,d_2} \Phi_{\pi_\Lambda,\tau_d} = 0$:

Lemma 4.2.

$$\begin{aligned}
(2)\quad & (r_2-k_2)\{\partial_1 - \frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1)\frac{1}{\mathrm{sh}(a_1^2)} \\
&- \frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2)\,\mathrm{cth}(a_1^2) + (k_2+1)\frac{\mathrm{sh}(a_1^2)}{D}\}c_{k_1,l_1;k_2,l_2} \\
&+ (k_2+1)(s_2-l_2+1)\frac{\mathrm{sh}(a_2^2)}{D}c_{k_1,l_1;k_2+1,l_2-1} \\
&+ 2(k_2+1)(r_1-k_1+1)\frac{\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2} \\
&+ 2(k_2+1)(s_1-l_1+1)\frac{\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2} = 0,
\end{aligned}$$

$$\begin{aligned}
(3)\quad & (k_2+1)\{\partial_2 - \frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1)\frac{1}{\mathrm{sh}(a_2^2)} \\
&- \frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2-4)\,\mathrm{cth}(a_2^2) - (r_2-k_2)\frac{\mathrm{sh}(a_2^2)}{D}\}c_{k_1,l_1;k_2+1,l_2-1} \\
&- (r_2-k_2)l_2\frac{\mathrm{sh}(a_1^2)}{D}c_{k_1,l_1;k_2,l_2} \\
&- 2(r_2-k_2)(k_1+1)\frac{\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1} \\
&- 2(r_2-k_2)(l_1+1)\frac{\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)}{D}c_{k_1,l_1+1;k_2,l_2-1} = 0,
\end{aligned}$$

$$\begin{aligned}
(4)\quad & (k_2+1)\{\partial_1 + \frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1)\frac{1}{\mathrm{sh}(a_1^2)} \\
&+ \frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2+4)\,\mathrm{cth}(a_1^2) + (r_2-k_2)\frac{\mathrm{sh}(a_1^2)}{D}\}c_{k_1,l_1;k_2+1,l_2-1} \\
&+ (r_2-k_2)l_2\frac{\mathrm{sh}(a_2^2)}{D}c_{k_1,l_1;k_2,l_2} \\
&+ 2(r_2-k_2)(k_1+1)\frac{\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1} \\
&+ 2(r_2-k_2)(l_1+1)\frac{\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)}{D}c_{k_1,l_1+1;k_2,l_2-1} = 0,
\end{aligned}$$

$$(5)\quad (r_2-k_2)\{\partial_2 + \frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1)\frac{1}{\mathrm{sh}(a_2^2)}$$

$$+\frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2)\operatorname{cth}(a_2^2)-(k_2+1)\frac{\operatorname{sh}(a_2^2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$-(k_2+1)(s_2-l_2+1)\frac{\operatorname{sh}(a_1^2)}{D}c_{k_1,l_1;k_2+1,l_2-1}$$
$$-2(k_2+1)(r_1-k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}$$
$$-2(k_2+1)(s_1-l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2}=0.$$

As for left equation systems, we have the following system:

$$\begin{aligned}
(6)\quad & (r_1-k_1)\{\partial_1-\frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2)\frac{1}{\operatorname{sh}(a_1^2)}\\
&-\frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1)\operatorname{cth}(a_1^2)+(k_1+1)\frac{\operatorname{sh}(a_1^2)}{D}\}c_{k_1,l_1;k_2,l_2}\\
&+(k_1+1)(s_1-l_1+1)\frac{\operatorname{sh}(a_2^2)}{D}c_{k_1+1,l_1-1;k_2,l_2}\\
&+2(k_1+1)(r_2-k_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}\\
&+2(k_1+1)(s_2-l_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}=0,
\end{aligned}$$

$$\begin{aligned}
(7)\quad & (k_1+1)\{\partial_2-\frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2)\frac{1}{\operatorname{sh}(a_2^2)}\\
&-\frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1-4)\operatorname{cth}(a_2^2)-(r_1-k_1)\frac{\operatorname{sh}(a_2^2)}{D}\}c_{k_1+1,l_1-1;k_2,l_2}\\
&-(r_1-k_1)l_1\frac{\operatorname{sh}(a_1^2)}{D}c_{k_1,l_1;k_2,l_2}\\
&-2(r_1-k_1)(k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2}\\
&-2(r_1-k_1)(l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1,l_1-1;k_2,l_2+1}=0,
\end{aligned}$$

$$\begin{aligned}
(8)\quad & (k_1+1)\{\partial_1+\frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2)\frac{1}{\operatorname{sh}(a_1^2)}\\
&+\frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1+4)\operatorname{cth}(a_1^2)+(r_1-k_1)\frac{\operatorname{sh}(a_1^2)}{D}\}c_{k_1+1,l_1-1;k_2,l_2}\\
&+(r_1-k_1)l_1\frac{\operatorname{sh}(a_2^2)}{D}c_{k_1,l_1;k_2,l_2}\\
&+2(r_1-k_1)(k_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2}\\
&+2(r_1-k_1)(l_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1-1;k_2,l_2+1}=0,
\end{aligned}$$

$$(9)\quad (r_1-k_1)\{\partial_2+\frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2)\frac{1}{\operatorname{sh}(a_2^2)}$$

$$
\begin{aligned}
&+\frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1)\operatorname{cth}(a_2^2)-(k_1+1)\frac{\operatorname{sh}(a_2^2)}{D}\}c_{k_1,l_1;k_2,l_2}\\
&-(k_1+1)(s_1-l_1+1)\frac{\operatorname{sh}(a_1^2)}{D}c_{k_1+1,l_1-1;k_2,l_2}\\
&-2(k_1+1)(r_2-k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}\\
&-2(k_1+1)(s_2-l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}=0.
\end{aligned}
$$

4.1. **Going up/down equations.** We can reduce the obtained equations to the following going up system (10), (11), (12), (13) as follows:

Lemma 4.3.

$$
\begin{aligned}
(10)\qquad &(r-k_2)\{\operatorname{cth}(a_1)\partial_1-(s-l_1-l_2)\operatorname{cth}^2(a_1)\\
&\quad-\frac{1}{2}(-u-k_1+k_2+l_1-l_2)+2(k_2+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}\\
&\quad+2(k_2+1)(r-k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}\\
&\quad=-2(k_2+1)(s-l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1,l_1;k_2+1,l_2-1}\\
&\quad-2(k_2+1)(s-l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2},
\end{aligned}
$$

$$
\begin{aligned}
(11)\qquad &(r-k_2)\{\operatorname{cth}(a_2)\partial_2-(s-l_1-l_2)\operatorname{cth}^2(a_2)\\
&\quad-\frac{1}{2}(u-k_1+k_2+l_1-l_2)-2(k_2+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}\\
&\quad-2(k_2+1)(r-k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}\\
&\quad=2(k_2+1)(s-l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1,l_1;k_2+1,l_2-1}\\
&\quad+2(k_2+1)(s-l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2},
\end{aligned}
$$

$$
\begin{aligned}
(12)\qquad &(r-k_1)\{\operatorname{cth}(a_1)\partial_1-(s-l_1-l_2)\operatorname{cth}^2(a_1)\\
&\quad-\frac{1}{2}(u+k_1-k_2-l_1+l_2)+2(k_1+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}\\
&\quad+2(k_1+1)(r-k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}\\
&\quad=-2(k_1+1)(s-l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1-1;k_2,l_2}\\
&\quad-2(k_1+1)(s-l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1},
\end{aligned}
$$

$$
\begin{aligned}
(13)\qquad &(r-k_1)\{\operatorname{cth}(a_2)\partial_2-(s-l_1-l_2)\operatorname{cth}^2(a_2)\\
&\quad-\frac{1}{2}(-u+k_1-k_2-l_1+l_2)-2(k_1+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}
\end{aligned}
$$

$$- 2(k_1+1)(r-k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$= 2(k_1+1)(s-l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1+1,l_1-1;k_2,l_2}$$
$$+ 2(k_1+1)(s-l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}.$$

Going down equations are as follows:

(14)
$$k_2\{\operatorname{cth}(a_1)\partial_1 + (s-l_1-l_2)\operatorname{cth}^2(a_1)$$
$$+\frac{1}{2}(-u-k_1+k_2+l_1-l_2) + 2(r-k_2+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$+ 2(r-k_2+1)(k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$= -2(r-k_2+1)(l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1,l_1;k_2-1,l_2+1}$$
$$- 2(r-k_2+1)(l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1+1;k_2-1,l_2},$$

(15)
$$k_2\{\operatorname{cth}(a_2)\partial_2 + (s-l_1-l_2)\operatorname{cth}^2(a_2)$$
$$+\frac{1}{2}(u-k_1+k_2+l_1-l_2) - 2(r-k_2+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$- 2(r-k_2+1)(k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$= 2(r-k_2+1)(l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1,l_1;k_2-1,l_2+1}$$
$$+ 2(r-k_2+1)(l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1,l_1+1;k_2-1,l_2},$$

(16)
$$k_1\{\operatorname{cth}(a_1)\partial_1 + (s-l_1-l_2)\operatorname{cth}^2(a_1)$$
$$+\frac{1}{2}(u+k_1-k_2-l_1+l_2) + 2(r-k_1+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$+ 2(r-k_1+1)(k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}$$
$$= -2(r-k_1+1)(l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1+1;k_2,l_2}$$
$$- 2(r-k_1+1)(l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1-1,l_1;k_2,l_2+1},$$

(17)
$$k_1\{\operatorname{cth}(a_2)\partial_2 + (s-l_1-l_2)\operatorname{cth}^2(a_2)$$
$$+\frac{1}{2}(-u+k_1-k_2-l_1+l_2) - 2(r-k_1+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$- 2(r-k_1+1)(k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}$$
$$= 2(r-k_1+1)(l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1-1,l_1+1;k_2,l_2}$$

$$+ 2(r-k_1+1)(l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1-1,l_1;k_2,l_2+1}.$$

To make equations more "symmetric", we consider (10) $\pm$ (11), etc, and rewrite them using p and t. Put

$$c_{k_1,l_1;k_2,l_2}(a) = (\operatorname{sh}(a_1)\operatorname{sh}(a_2))^{|s-l_1-l_2|}(\operatorname{ch}(a_1)\operatorname{ch}(a_2))^{-(r+s+2)/2}\tilde{c}_{k_1,l_1;k_2,l_2}(a).$$

In the following, we assume that $0 \leq l_1 + l_2 \leq s$. We remark that

$$\begin{aligned} 2\partial_p &= \operatorname{cth}(a_1)\partial_1 + \operatorname{cth}(a_2)\partial_2, \\ 4\partial_t &= \operatorname{cth}(a_1)\partial_1 - \operatorname{cth}(a_2)\partial_2. \end{aligned}$$

Then, we have

Lemma 4.4.

$$\begin{aligned}
&(18) \qquad (r-k_2)(\partial_p - l_1)\tilde{c}_{k_1,l_1;k_2,l_2} = (k_2+1)(s-l_2+1)p\tilde{c}_{k_1,l_1;k_2+1,l_2-1} \\
&\qquad\qquad + (k_2+1)(s-l_1+1)\tilde{c}_{k_1,l_1-1;k_2+1,l_2}, \\
&(19) \qquad (r-k_2)\left(2\partial_t + \frac{u}{2} + (k_2+1)\frac{t+1}{t-1}\right)\tilde{c}_{k_1,l_1;k_2,l_2} \\
&\qquad\qquad + 2(k_2+1)(r-k_1+1)z_-(t)^{-1}\tilde{c}_{k_1-1,l_1;k_2+1,l_2} \\
&\qquad\qquad = (k_2+1)(s-l_2+1)z_-(t)^{-1}(2-pz_+)\tilde{c}_{k_1,l_1;k_2+1,l_2-1} \\
&\qquad\qquad + (k_2+1)(s-l_1+1)z_-(t)^{-1}(z_+-2p)\tilde{c}_{k_1,l_1-1;k_2+1,l_2}, \\
&(20) \qquad k_2\{(p^2 - z_+(t)p+1)(\partial_p + l_1 - s) \\
&\qquad\qquad + (s-l_1-l_2)(2p-z_+(t))p\}\tilde{c}_{k_1,l_1;k_2,l_2} \\
&\qquad\qquad = (r-k_2+1)(l_2+1)p\tilde{c}_{k_1,l_1;k_2-1,l_2+1} \\
&\qquad\qquad + (r-k_2+1)(l_1+1)\tilde{c}_{k_1,l_1+1;k_2-1,l_2}, \\
&(21) \qquad (r-k_1)(\partial_p - l_2)\tilde{c}_{k_1,l_1;k_2,l_2} = (k_1+1)(s-l_1+1)p\tilde{c}_{k_1+1,l_1-1;k_2,l_2} \\
&\qquad\qquad + (k_1+1)(s-l_2+1)\tilde{c}_{k_1+1,l_1;k_2,l_2-1}, \\
&(22) \qquad (r-k_1)\left(2\partial_t - \frac{u}{2} + (k_1+1)\frac{t+1}{t-1}\right)\tilde{c}_{k_1,l_1;k_2,l_2} \\
&\qquad\qquad + 2(k_1+1)(r-k_2+1)z_-(t)^{-1}\tilde{c}_{k_1+1,l_1;k_2-1,l_2} \\
&\qquad\qquad = (k_1+1)(s-l_1+1)z_-(t)^{-1}(2-pz_+)\tilde{c}_{k_1+1,l_1-1;k_2,l_2} \\
&\qquad\qquad + (k_1+1)(s-l_2+1)z_-(t)^{-1}(z_+-2p)\tilde{c}_{k_1+1,l_1;k_2,l_2-1}, \\
&(23) \qquad k_1\{(p^2 - z_+(t)p+1)(\partial_p + l_2 - s) \\
&\qquad\qquad + (s-l_1-l_2)(2p-z_+(t))p\}\tilde{c}_{k_1,l_1;k_2,l_2} \\
&\qquad\qquad = (r-k_1+1)(l_1+1)p\tilde{c}_{k_1-1,l_1+1;k_2,l_2} \\
&\qquad\qquad + (r-k_1+1)(l_2+1)\tilde{c}_{k_1-1,l_1;k_2,l_2+1}.
\end{aligned}$$

As we know, the equations (21), (22) and (23) can be obtained by flipping indices 1 and 2:

Remark 4.5. We have similar equations when $s \leq l_1 + l_2 \leq 2s$.

5. Solution for the holonomic system: the main theorem

5.1. Separation of variables. We treat the case when $l_1 + l_2 \leq s$.

Proposition 5.1. *Write $M = (k_1, l_1; k_2, l_2)$. Then $\tilde{c}_M$ can be written in the form of "separation of variables":*

$$\tilde{c}_M(a) = \sum_{i=0}^{l_1+l_2} (-1)^{r-k_1-l_1} \binom{r}{k_1} \binom{r}{k_2} p^{l_1+l_2-i} s_{M;i}(t).$$

We can prove it by induction on $l_1 + l_2$. Assume that $l_1 = l_2 = 0$. By (18), we have,

$$\partial_p \tilde{c}_{k_1,0;k_2,0} = 0,$$

so that actually we can put $s_{(k_1,0;k_2,0),0}(t) := \tilde{c}_{k_1,0;k_2,0}(a)$. Next assume that $l_1 + l_2 > 0$. If $l_1 \neq l_2$ and $k_1 < r$, $k_2 < r$, then $(18)/(r-k_2) - (21)/(r-k_1)$ shows the assertion. Otherwise, we can assume $k_2 \neq r$. Consulting (18), we readily prove the formula.

According to Proposition 5.1, we can rewrite the difference equations of Lemma 4.4 in terms of p and t. Comparing the coefficients as a polynomial of p, we have the following.

Lemma 5.2. 1. *If $0 \leq k_2 < r$, then,*

$$(l_2 - i) s_{(k_1,l_1;k_2,l_2),i} = (s - l_2 + 1) s_{(k_1,l_1;k_2+1,l_2-1),i} - (s - l_1 + 1) s_{(k_1,l_1-1;k_2+1,l_2),i-1}, \tag{24}$$

$$\begin{aligned}\left(2\partial_t + \frac{u}{2} + (k_2+1)\frac{t+1}{t-1}\right) s_{(k_1,l_1;k_2,l_2),i} - \frac{2k_1}{z_-(t)} s_{(k_1-1,l_1;k_2+1,l_2),i} \\ = (s - l_2 + 1)\left(\frac{2}{z_-(t)} s_{(k_1,l_1;k_2+1,l_2-1),i-1} - \frac{t+1}{t-1} s_{(k_1,l_1;k_2+1,l_2-1),i}\right) \\ - (s - l_1 + 1)\left(\frac{t+1}{t-1} s_{(k_1,l_1-1;k_2+1,l_2),i-1} - \frac{2}{z_-(t)} s_{(k_1,l_1-1;k_2+1,l_2),i}\right),\end{aligned} \tag{25}$$

$$\begin{aligned}l_2 s_{(k_1,l_1;k_2,l_2),i+1} - (l_1+1) s_{(k_1,l_1+1;k_2,l_2-1),i} \\ = (s - l_2 - i) s_{(k_1,l_1;k_2+1,l_2-1),i+1} - (l_1 - i) z_+(t) s_{(k_1,l_1;k_2+1,l_2-1),i} \\ + (2l_1 + l_2 - s - i) s_{(k_1,l_1;k_2+1,l_2-1),i-1}.\end{aligned} \tag{26}$$

2. *If $0 \leq k_1 < r$, then,*

$$(l_1 - i) s_{(k_1,l_1;k_2,l_2),i} = (s - l_1 + 1) s_{(k_1+1,l_1-1;k_2,l_2),i} - (s - l_2 + 1) s_{(k_1+1,l_1;k_2,l_2-1),i-1}, \tag{27}$$

$$\begin{aligned}\left(2\partial_t - \frac{u}{2} + (k_1+1)\frac{t+1}{t-1}\right) s_{(k_1,l_1;k_2,l_2),i} - \frac{2k_2}{z_-(t)} s_{(k_1+1,l_1;k_2-1,l_2),i} \\ = (s - l_1 + 1)\left(\frac{2}{z_-(t)} s_{(k_1+1,l_1-1;k_2,l_2),i-1} - \frac{t+1}{t-1} s_{(k_1+1,l_1-1;k_2,l_2),i}\right) \\ - (s - l_2 + 1)\left(\frac{t+1}{t-1} s_{(k_1+1,l_1;k_2,l_2-1),i-1} - \frac{2}{z_-(t)} s_{(k_1+1,l_1;k_2,l_2-1),i}\right),\end{aligned} \tag{28}$$

$$\begin{aligned}l_1 s_{(k_1,l_1;k_2,l_2),i+1} - (l_2+1) s_{(k_1,l_1-1;k_2,l_2+1),i} \\ = (s - l_1 - i) s_{(k_1+1,l_1-1;k_2,l_2),i+1} - (l_2 - i) z_+(t) s_{(k_1+1,l_1-1;k_2,l_2),i} \\ + (2l_2 + l_1 - s - i) s_{(k_1+1,l_1-1;k_2,l_2),i-1}.\end{aligned} \tag{29}$$

5.2. **Expression of peripheral entries using Gaussian hypergeometric functions.** First assume that $l_1 = l_2 = 0$. We simply write $s_{k_1,k_2} = s_{(k_1,0;k_2,0),0}$. By (19) and (22), we have

$$(r-k_2)\left(\partial_t + \frac{u}{4} + \frac{k_2+1}{2}\frac{t+1}{t-1}\right)s_{k_1;k_2} + \frac{(k_2+1)(r-k_1+1)}{z_-(t)}s_{k_1-1;k_2+1} = 0,$$
$$(r-k_1+1)\left(\partial_t - \frac{u}{4} + \frac{k_1}{2}\frac{t+1}{t-1}\right)s_{k_1-1;k_2+1} + \frac{k_1(r-k_2)}{z_-(t)}s_{k_1;k_2} = 0.$$

Eliminating $s_{k_1-1;k_2+1}$, we have

$$\left\{\left(\partial_t - \frac{u}{4} + \frac{k_1+1}{2}\frac{t+1}{t-1}\right)\left(\partial_t + \frac{u}{4} + \frac{k_2+1}{2}\frac{t+1}{t-1}\right) - k_1(k_2+1)z_-(t)^{-2}\right\}s_{k_1;k_2} = 0.$$

Considering $r+s = k_1+k_2$, we have

$$\left(\partial_t^2 + \frac{r+s+2}{2}\frac{t+1}{t-1}\partial_t + \frac{u(k_1-k_2)}{8}\frac{t+1}{t-1} + \frac{(r+s+2)^2-(k_1-k_2)^2-u^2}{16}\right)s_{k_1;k_2} = 0$$

and its Riemann's P scheme is:

$$P\begin{bmatrix} 0 & 1 & \infty \\ \frac{r+s+2}{4} - \frac{k_1-k_2+u}{4} & 0 & \frac{r+s+2}{4} + \frac{k_1-k_2-u}{4} \\ \frac{r+s+2}{4} + \frac{k_1-k_2+u}{4} & -(r+s+1) & \frac{r+s+2}{4} - \frac{k_1-k_2-u}{4} \end{bmatrix}.$$

In general, let $\Phi(m_1,m_2) = \Phi(m_1,m_2;u;t)$ be a regular function around 1 having the P-scheme

$$P\begin{bmatrix} 0 & 1 & \infty \\ \frac{m_1+m_2+2}{4} - \frac{m_1-m_2+u}{4} & 0 & \frac{m_1+m_2+2}{4} - \frac{m_1-m_2-u}{4} \\ \frac{m_1+m_2+2}{4} + \frac{m_1-m_2+u}{4} & -(m_1+m_2+1) & \frac{m_1+m_2+2}{4} + \frac{m_1-m_2-u}{4} \end{bmatrix}$$

with condition $\Phi(m_1,m_2;u;1) = \binom{r+s}{m_1}^{-1}$. We also write $\Phi(m) = \Phi(m, r+s-m)$ for simplicity. Then it follows $s_{(k_1,0;k_2,0),0} = c_0\Phi(k_1,k_2)$.

5.3. **Reduction of general coefficients $s_{M,i}$.** To describe general solutions, we introduce the notion of height and bias. Write $M = (k_1,l_1;k_2,l_2)$ as before. Define $h = h(M,i) = \min(i,l_1,l_2,l_1+l_2-i)$ and

$$b = b(M,i) = \begin{cases} 0 & i \le \min(l_1,l_2), \\ \operatorname{sgn}(l_2-l_1)(i-\min(l_1,l_2)) & \min(l_1,l_2) \le i \le \max(l_1,l_2), \\ l_2-l_1 & \max(l_1,l_2) \le i. \end{cases}$$

Then we have,

Proposition 5.3.

$$s_{M,i} = \sum_{j=b-h}^{b+h} Q_j(-z_+)\Phi(k_1+l_1+j) \tag{30}$$

for a polynomial $Q_j(t) = Q_j(M,i;t)$ which is actually independent of the choice of r, k_1 and k_2. The degree of Q_j is equal to $h - |j-b|$ and it follows

$$Q_j(-z_+) = (-1)^{\deg Q_j}Q_j(z_+).$$

5.4. **Polynomials $Q_j(z_+)$.** The remaining paper deals with the determination of the polynomial Q_j. We can deduce the difference equations of Q_j equivalent to (26). Proposition 5.3 says that Q_j is in the form

$$Q_j(z_+) = \sum_{m \geq 0} \tilde{\beta}_m(M, i, j) z_+^{h-|j|-2m}.$$

For simplicity, we put $\tilde{\beta}_m(M, i, j) = \binom{s}{l_1}\binom{s}{l_2}\beta_m(M, i, j)$. Comparing the coefficient of $z_+^{h-|j|-2m}$, we see that our difference equations become as follows: If $j \geq 0$, then,

$$\begin{aligned}(31) \qquad (s-i+1)i\beta_m(M,i,j) &= (s-l_1)l_2\beta_m(M+(0,1;0,-1), i-1, j-1) \\ &+ l_1(s-l_2-i+1)\beta_{m-1}(M+(0,-1;1,0), i-1, j+1) \\ &+ (l_1-i+1)l_2\beta_m(M+(0,0;1,-1), i-1, j) \\ &+ (2l_1+l_2-s-i+1)l_2\beta_{m-1}(M+(0,0;1,-1), i-2, j).\end{aligned}$$

If $j < 0$, then,

$$\begin{aligned}(32) \qquad (s-i+1)i\beta_m(M,i,j) &= (s-l_1)l_2\beta_{m-1}(M+(0,1;0,-1), i-1, j-1) \\ &+ l_1(s-l_2-i+1)\beta_m(M+(0,-1;1,0), i-1, j+1) \\ &+ (l_1-i+1)l_2\beta_m(M+(0,0;1,-1), i-1, j) \\ &+ (2l_1+l_2-s-i+1)l_2\beta_{m-1}(M+(0,0;1,-1), i-2, j).\end{aligned}$$

The solution can be expressed as follows:

Proposition 5.4. *Assume that* $0 \leq l_1 + l_2 \leq s$. *Then,*

$$\beta_m = \alpha(m; i, |j|)\binom{l_1}{i-j_+-m}\binom{l_2}{i+j_--m}\sum_{n=0}^{m}\binom{s-l_1}{j+m-n}\binom{s-l_2}{m-n}\binom{s-i+n}{n}$$

for

$$\alpha(m; i, j) = \binom{i-j-m}{m}\binom{i}{m}^{-1}\binom{s}{i}^{-1}, \quad j_+ = \begin{cases} j & (j \geq 0), \\ 0 & (j < 0) \end{cases}$$

and $j_- = (-j)_+$.

We can check that each $\beta_m(M, i, j)$ fits the definition of h as $\beta_m(M, i, j)$ is nonzero if and only if $i - |j| - 2m \geq 0$, $l_1 - i + j_+ + m \geq 0$ and $l_2 - i - j_- + m \geq 0$.

Main Theorem 5.5. *Let* π_Λ *be a middle discrete series representation with* $\Lambda = [r-1, s+1; u] \in \Xi_{\mathrm{III}}$, *and* τ_d *the minimal* K*-type of* π_Λ *with* $d = [r, s; u]$. *For a* (τ_d, τ_d^*)*-matrix coefficient* $\Phi_{\pi,\tau}$, *put*

$$\Phi_{\pi,\tau}(a) = \sum_{k_1, l_1; k_2, l_2} c_{k_1,l_1;k_2,l_2}(a) f_{k_1,l_1;k_2,l_2}.$$

Then, $c_M(a)$ $(M = (k_1, l_1;\, k_2, l_2))$ *is given by the following:*

1. *Suppose that* $l_1 + l_2 \leq s$. *The matrix coefficients* $c_{k_1,l_1;k_2,l_2}(a_1, a_2)$ *can be expressed as follows*:

$$
\begin{aligned}
c_M(a_1, a_2) &= c_0(-1)^{r-k_1-l_1}(\mathrm{sh}(a_1)\,\mathrm{sh}(a_2))^{s-l_1-l_2} \\
&\times \sum_{i=0}^{l_1+l_2} (\mathrm{ch}(a_1)\,\mathrm{ch}(a_2))^{-(r+s+2)/2+l_1+l_2-i} \binom{r}{k_1}\binom{r}{k_2}\binom{s}{l_1}\binom{s}{l_2} \\
&\times \sum_{j=b-h}^{b+h} (-1)^{h-|j-b|} \sum_{\mu=0}^{[\frac{i-|j-b|}{2}]} \beta_\mu(M, i, j) \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)} + \frac{\mathrm{ch}(a_2)}{\mathrm{ch}(a_1)}\right)^{h-|j-b|-2\mu} \\
&\times \Phi\left(k_1 + l_1 + j, k_2 + l_2 - j; u; \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)}\right)^2\right).
\end{aligned}
$$

2. *Suppose that* $s < l_1+l_2 \leq 2s$. *Define* $M^\wedge = (r-k_1, s-l_1; r-k_2, s-l_2)$, $b^\wedge = b(M^\wedge, i)$ *and* $h^\wedge = h(M^\wedge, i)$. *Then,*

$$
\begin{aligned}
c_M(a_1, a_2) &= c_0(-1)^{r-k_1-l_1}(\mathrm{sh}(a_1)\,\mathrm{sh}(a_2))^{-s+l_1+l_2} \\
&\times \sum_{i=0}^{l_1+l_2} (\mathrm{ch}(a_1)\,\mathrm{ch}(a_2))^{-(r+s+2)/2+2s-l_1-l_2-i} \binom{r}{k_1}\binom{r}{k_2}\binom{s}{l_1}\binom{s}{l_2} \\
&\times \sum_{j=b^\wedge-h^\wedge}^{b^\wedge+h^\wedge} (-1)^{h^\wedge-|j-b^\wedge|} \sum_{m=0}^{[\frac{i-|j-b^\wedge|}{2}]} \beta_m(M^\wedge, i, j) \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)} + \frac{\mathrm{ch}(a_2)}{\mathrm{ch}(a_1)}\right)^{h^\wedge-|j-b^\wedge|-2m} \\
&\times \Phi\left(k_2 + l_2 + j, k_1 + l_1 - j; -u; \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)}\right)^2\right).
\end{aligned}
$$

Remark 5.6. We can determine the unique unknown constant c_0 by using the normalization condition, *i.e.*, by specification of the value of Φ at the identity of G.

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